A Degree of Controllability Definition: Fundamental Concepts and Application to Modal Systems

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Starting from basic physical considerations, this paper develops a concept of the degree of controllability of a control system, and then develops numerical methods to generate approximate values of the degree of controllability for any linear time-invariant system. In many problems, such as the control of future, very large, flexible spacecraft and certain chemical process control problems, the question of how to choose the number and locations of the control system actuators is an important one. The results obtained here offer the control system designer a tool which allows him to rank the effectiveness of alternative actuator distributions, and hence to choose the actuator locations on a rational basis. The degree of controllability is shown to take a particularly simple form when the dynamic equations of a satellite are in second-order modal form. The degree of controllability concept has still other fundamental uses—it allows one to study the system structural relations between the various inputs and outputs of a linear system, which has applications to decoupling and model reduction.

Introduction

IN the last few years a large number of potential future satellite projects have been identified which require spacecraft of unprecedented size, and hence unprecedented flexibility. The attitude control problem for such a spacecraft is best characterized as simultaneous pointing control and shape control of the vehicle. In order to achieve shape control, or, equivalently, control of the various modes of oscillation of the flexible structure, it will be necessary to distribute actuators throughout the vehicle. This paper has been motivated by the important questions, how should the number and locations of actuators be chosen in order to best control the flexible spacecraft? The results obtained have much wider applicability, not only to other actuator placement problems, but also to the study of the importanced or lack of importance of each control variable to the various outputs of the system, with resulting implications to decoupling or model reduction in systems.

The concept of controllability in modern control theory is a binary concept; either a system is controllable or it is uncontrollable. Starting from a set of actuator locations which produce an uncontrollable system, but for which the number of actuators is sufficient to produce controllability, it will usually be the case that moving one of the actuators by a distance $\epsilon > 0$ can produce a controllable system, no matter how small the ϵ. One expects that for a small ϵ, even though technically the system is controllable, in some sense it will not be very controllable.

It is the purpose of this paper to generate, starting from basic physical considerations, a precise definition of this concept. The definition obtained is quite natural and has the important property that it incorporates the physical reality of actuator saturation limitations in a simple and fundamental way. The basic concepts for the definition are generated in this work and a simple method of calculating the approximate degree of controllability is developed for the case when the system matrix has only distinct eigenvalues. The concept is then applied to modal systems and found to give particularly simple results. A companion paper, Ref. 3, shows that special techniques are required for the approximate degree of controllability when the system matrix contains repeated eigenvalues, (these results are also given in Ref. 4) and then applies the method to actuator placement for attitude and shape control of a simple flexible spacecraft. Of course, attitude control necessarily involves repeated eigenvalues.

Definition of the Degree of Controllability

Let us consider any general linear time-invariant system in state variable form

$$\dot{x}^*(t) = \mathcal{A}x^*(t) + \mathcal{B}u^*(t) \tag{1}$$

where $x^* \epsilon R^n$ and $u^* \epsilon R^m$. It should be noted that although we focus our attention on this system, the degree of controllability definition which we adopt is also applicable to more general systems of the form $\dot{x}^*(t) = f(x^*, u^*, t)$ having a solution $x^*(t) \equiv 0$ $[f(0,0,t) = 0]$.

The standard textbook test for controllability is the rank of the matrix $\mathcal{Q}^* = [\mathcal{B} \; \mathcal{A}\mathcal{B} \; \ldots \; \mathcal{A}^{n-1}\mathcal{B}]$. A simplistic approach to defining a degree of controllability is to determine how near $\mathcal{Q}^*$ is to being rank deficient by looking at the minimum eigenvalue of $\mathcal{Q}^*\mathcal{Q}^{*T}$. It is instructive to examine such simplistic approaches, since their failures highlight the characteristics that a workable definition must have. Five apparent difficulties with the above candidate definition follow.

1) The degree of controllability is affected by a transformation of coordinates (since the eigenvalues of $\mathcal{Q}^*\mathcal{Q}^{*T}$ are not invariant under changes in state variable representation). A certain type of coordinate system dependence will in fact be needed, but at this point the need for, and the nature of, such dependence is not obvious.

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2) This candidate definition satisfies the basic requirement that the degree of controllability is zero when the system is uncontrollable, but it is not immediately clear what other physical meaning can be attached to the size of the eigenvalues of $\mathcal{Q}^*\mathcal{Q}^{*T}$.

3) Different control objectives should influence the choice of the control system design, and hence the degree of controllability of a candidate design should be keyed to the objective involved. In a large class of problems (regulator problems), the equilibrium solution $x^*=0$ to Eq. (1) is of primary importance, and the control objective is to return x^* to zero after a disturbance. Certainly this objective is much easier to accomplish for a stable system than an unstable system. It is not clear how the actual control objective could be reflected in the above definition, and it is not clear that the dependence on stability is reflected properly.

4) The candidate definition does not involve a dependence on the amount of time T allotted to accomplish the control task. It can be much easier to control the system state in some directions in the state space at one time than at another time, so the degree of controllability should depend on T.

5) It is not clear that the amount of control effort needed to accomplish the control task is reflected in this definition. In the satellite described above where one actuator has been moved by a small amount ϵ to produce controllability, one expects the "weak controllability" of the system to be manifested in the need for very large control actions to accomplish certain small changes in the state, and hence the control effort required is of fundamental importance in making a definition.

It is clear that some type of limitation or standardization of the control effort must be included in the definition. Consider a standardization which restricts the control to a unit impulse, and consider systems with $\mathcal{A}$ in diagonal form and with u^* a scalar. For distinct eigenvalues, the system is controllable if none of the elements $\mathcal{B}_i$ of the column matrix $\mathcal{B}$ are zero. Furthermore, these components indicate how far a unit impulse control will move each state component instantaneously, so one might suggest the $\min_i |\mathcal{B}_i|$ as a degree of controllability. Here we are trying to generalize a second standard test for controllability to obtain a degree of controllability definition. Among the apparent difficulties with this candidate definition is the fact that the control actions are so restricted that the components of the state cannot be affected independently. The control of all states by a single control u^* relies on the differences in the dynamic behaviors of the states. Objections 3 and 4 above also apply.

Both of these candidate definitions have difficulties—they do not appear to include the effects of all pertinent variables. Hence it will be necessary to build the definition from more fundamental considerations. Ironically, when this is completed and interpreted properly, in certain special cases the degree of controllability definition will be a weighted version of the second candidate definition above (and by employing a different approach one can obtain a weighted version of the first candidate definition when altered to start from the controllability Gramian).

It is now evident that the definition of the degree of controllability, besides being in some sense a measure of how easy it is for the controller to control the system, must in some way handle five things:

1) It must have the property that the degree of controllability is zero when the system is uncontrollable.

2) The system stability properties must somehow be represented.

3) It must somehow consider dependence on total time T.

4) It must standardize or restrict the control effort in some way.

5) The control objective must be reflected in the definition.

For purposes of this paper, the control objective will be to return x^* to zero after a disturbance (the regulator problem), since this is the most common attitude and shape control objective for flexible spacecraft. Concerning the standardization of the control effort, we will require that the control components satisfy $|u_i| \leq 1$ for $i=1,2,3,\ldots,m$, which represents realistic physical limitations of the actuator capabilities. Note that the use of 1 as the bound for all control components implies normalizing each component of u^* to produce a new control vector u, and adjusting the $\mathcal{B}$ matrix to produce a new matrix B.

Controllability requires the existence of a control function which can transfer any initial state to any final state in finite time. With our more limited control objective, the degree of controllability should be related to the volume of initial system states (or states resulting from disturbances) which can be returned to the desired state $x^*=0$ in time T using the bounded controls. Consider the nature of this volume in more detail. In an uncontrollable system there will be at least one direction in the state space for which initial conditions in this direction cannot be returned to the origin, and the volume will lose one or more dimensions. For a controllable system whose parameters are such that it is nearly uncontrollable, only initial conditions very close to $x^*=0$ along the abovementioned direction could be returned to the origin in time T using the bounded controls. Hence we will generate a definition of the degree of controllability based on the minimum distance from the origin to a normalized state that cannot be brought to the origin in time T. More loosely it is the minimum disturbance from which the system cannot recover in time T.

The coordinates of a state space will very rarely all have the same physical units, and hence it is clear that comparing distances in the state space will require that each coordinate be made unitless by normalization. How should one choose the normalization to use? Recognize that when comparing two controller designs for controlling the same dynamic system, the needed minimum distance for the designs will usually correspond to different directions in the state space. Hence ranking of the degree of controllability of the two systems will depend on comparison of distances in different directions, and this implies that we must normalize the state vector such that the need for recovery from an initial unit displacement away from the origin is considered to be equal for all displacement directions. Denote the normalized state vector which has this property by x. The system equations in terms of this state are written as

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad |u_i| \leq 1 \qquad i=1,2,3,\ldots,m \tag{2}$$

In order to define the degree of controllability, it is necessary to be fully specific not only about the objective of keeping $x=0$, but also about the relative importance of keeping each component of x near zero.

Relative to the normalized system (2) we are now ready to make the following definitions:

Definition 1: The *recovery region for time T* for normalized system (2) is the set

$$\mathcal{R} = \{x(0) \mid \exists u(t), \quad t\in[0,T], \quad |u_i(t)| \leq 1 \quad \text{for} \quad i=1,2,\ldots,m \qquad \ni x(T)=0\}$$

Definition 2: The *degree of controllability in time T of the $x=0$ solution* of normalized system (2) is defined as

$$\rho = \inf\|x(0)\| \qquad \forall x(0) \notin \mathcal{R}$$

where $\|\cdot\|$ represents the Euclidian norm.

Thus the recovery region identifies all of the initial conditions (or disturbed states) which can be returned to the origin in time T using the bounded controls. And the degree of controllability is a scalar measure of the size of the region, where the scalar is chosen as the shortest distance from the origin to an initial state which cannot be returned to the origin in time T.

The degree of controllability, as defined, is keyed to the state vector x employed. No transformations of coordinates can be allowed once the normalization has been specified, unless the norm used in the definition is adjusted to compensate for the resulting distortion of the state space.

Note the following property of the recovery region:

Remark: The recovery region $\mathcal{R}$ for system (2) is the same as the set of reachable states for system (2) when t runs backwards from T to 0 with $x(T)=0$. Hence the dependence of the degree of controllability on system stability is evident.

It should be pointed out that although Definition 2 incorporates all the properties which were identified as necessary in the definition of the degree of controllability, it is not necessarily unique in doing so. For example, a standardization of the control effort in terms of energy can also be employed, but the present approach has the advantage that the practical reality of saturation constraints on the controls is included. Still other possibilities are discussed at the conclusion of this paper.

If one is concerned with a system output vector $y=Cx$ rather than the full state vector x, the concepts developed above are equally applicable. The normalization is made on the y components used as components of a transformed state vector, the remaining components of which are normalized so that they do not influence the degree of controllability. This approach allows one to study the structure of the input/output relations of the system. One can determine by the methods of the following sections, the degree of controllability of any output component or set of components from any control or set of control variables. Such concepts are useful for model reduction and system decoupling.

Concepts for Approximating the Recovery Region

In order to make the definition of the degree of controllability useful, it is necessary to develop a simple algorithm to generate at least an approximation to the distance ρ, and this necessitates approximating the recovery region $\mathcal{R}$. This section introduces certain concepts useful for making such an approximation, and in the process a method is developed which could be used on scalar control problems (although the method of the next section would generally be preferable).

Note that

$$x(T)=\phi(T,0)x(0)+\phi(T,0)\int_0^T \phi(0,t)Bu(t)\,\mathrm{d}t$$

where ϕ is the state transition matrix for Eq. (2). We are concerned with sending the system to the origin so that $x(T)=0$. Then the negative of the initial state which reaches the origin in time T using control $u(t)$ is given by

$$\delta=\int_0^T \phi(0,t)Bu(t)\,\mathrm{d}t \tag{3}$$

The recovery region is the set of all δ's which can be generated by admissible controls. Note that the recovery region will be symmetric in the sense that if admissible control $u(t)$ transfers $x(0)=-\delta$ to the origin in time T, then $-u(t)$ which is also an admissible control transfers $-x(0)=\delta$ to the origin in time T. Also, the set of reachable states for time T from $x(0)=0$ is the set of all δ^* where

$$\delta^*=\phi(T,0)\int_0^T \phi(0,t)Bu(t)\,\mathrm{d}t=\phi(T,0)\delta \tag{4}$$

For an asymptotically stable system, $\phi(T,0)$ is a contraction mapping so that the set of reachable states is smaller than the recovery region. When the system is unstable there will be at least one direction in the state space for which the boundary of the set of reachable states is outside the boundary of the recovery region.

By the Caley-Hamilton theorem, the state transition matrix can be written as

$$\phi(0,t)=e^{-At}=\sum_{\alpha=0}^{n-1}\psi_\alpha(t)A^\alpha \tag{5}$$

where the ψ_α are scalar functions of time. Partition the B matrix into column matrices b_j, and define the following matrices:

$$B=[b_1\ b_2\ b_3 \ldots b_m] \tag{6}$$

$$\psi^T=[\psi_0\ \psi_1\ \psi_2 \ldots \psi_{n-1}] \tag{7}$$

$$\mathcal{Q}=[B\ AB\ A^2B \ldots A^{n-1}B] \tag{8}$$

$$\mathcal{Q}_\beta=[b_\beta\ Ab_\beta\ A^2b_\beta \ldots A^{n-1}b_\beta] \tag{9}$$

Then δ can be represented in the following alternative forms:

$$\delta=\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{m}\left\{\int_0^T \psi_\alpha(t)u_\beta(t)\,\mathrm{d}t\right\}A^\alpha b_\beta \tag{10}$$

$$\delta=\sum_{\beta=1}^{m}\int_0^T [\psi_0 b_\beta+\psi_1 Ab_\beta+\ldots+\psi_{n-1}A^{n-1}b_\beta]u_\beta\,\mathrm{d}t \tag{11}$$

$$\delta=\sum_{\beta=1}^{m}\int_0^T [\mathcal{Q}_\beta\psi]u_\beta\,\mathrm{d}t \tag{12}$$

For the purposes of illustrating certain concepts, let us restrict ourselves to the case of a scalar control so that the summations over β as well as the β subscripts in the above can be dropped, and B is a column matrix b. Also let $n=2$ for simplicity. Suppose the recovery region is as shown by Region I in Fig. 1. The maximum x_1 component of any state in the recovery region is obtained by using the control u equal to minus the signum function of the first component of the vector $[\mathcal{Q}\psi]$ in Eq. (12), since this maximizes the x_1 component of the integrand at each time t. The right-hand side (and left-hand side) of the rectangle enclosing this recovery region in Fig. 1 can thus be found by integrating the first component in Eq. (12) using this control. If desired, the point at which the recovery region touches this side is obtained by integrating the second component of Eq. (12) using this control. The top and bottom of the rectangle are found similarly.

The rectangle obtained in this manner might be considered an approximation to the recovery region, and then the shortest distance from the origin to one of the sides might be considered an approximation, $\bar{\rho}$, to the degree of controllability, $\rho=\rho_{\mathrm{I}}$. Note that this necessarily produces a $\bar{\rho}$ which is an upper bound for the degree of controllability. In some cases, this approximation is a tight one, but often it is not. Suppose the recovery region was Region II of Fig. 1. This corresponds to a system which has a much poorer degree of controllability, $\rho=\rho_{\mathrm{II}}$, yet the approximation $\bar{\rho}$ remains the same. In fact, suppose that $\rho_{\mathrm{II}}\to 0$ in such a way that Region II degenerates to a line forming a diagonal of the rectangle in Fig. 1. Then the system is uncontrollable, but the approximation $\bar{\rho}$ still predicts a good degree of controllability. Hence this approximation is rejected.

For the case of a scalar control being considered, this shortcoming can be eliminated by using δ as expressed in Eq. (10) and maximizing components along $A^\alpha b$. The control

$$u(t)=-\mathrm{sgn}[\psi_\alpha(t)] \tag{13}$$

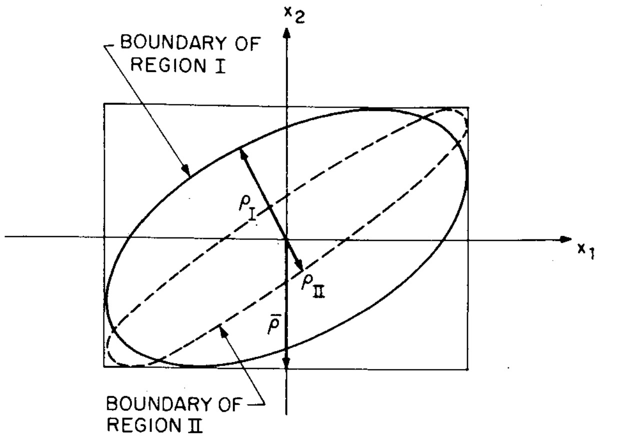

Fig. 1 Recovery region and its rectangular approximation.

extremizes the coefficient of the vector $A^\alpha b$ in Eq. (10). It will simultaneously produce some components along the other vectors $A^\gamma b$ for $\gamma \neq \alpha$. This is a maximization of a component of the vector δ, but it is a component as seen in a nonorthogonal set of coordinates. Hence the upper bounds obtained in the various directions define a parallelogram (more generally an n-dimensional parallelepiped) which can be considered as an approximation to the recovery region, as shown in Fig. 2. As before, there is some point on each side of the parallelogram which is in the recovery region, but no point outside the parallelogram is in the region.

The minimum distance to a side of the parallelogram, i.e., the minimum perpendicular distance to a side, is an approximation ρ^* to the degree of controllability, ρ. When the system becomes uncontrollable, the columns of $\mathcal{Q}$ become linearly dependent, and hence the perpendicular distance to one of the sides becomes zero. This means that this ρ^* has the essential property that $\rho^* = 0$ whenever $\rho = 0$.

We conclude that for the scalar control case we have a viable method of approximating the degree of controllability. A simple method will be presented in a later section to determine the needed minimum perpendicular distance.

This approximation is still an upper bound, and it can be improved, in fact, made arbitrarily good, by considering more directions in the state space. Let e be any desired unit vector expressed as a column matrix of components. By examining Eq. (12), the state in the recovery region having a maximum component along the direction e is obtained using the control

$$u = -\operatorname{sgn}[e^T \mathcal{Q} \psi] \tag{14}$$

and hence no points in the recovery region lie beyond the line perpendicular to e and a distance

$$\int_0^T |e^T \mathcal{Q} \psi| \, \mathrm{d}t \tag{15}$$

from the origin (but at least one point in the recovery region lies on the line). Figure 3 illustrates how use of three e's (e_I, e_{II}, e_{III}) identifies three tangents to the recovery region, and when taken together they begin to approximate the region boundary. Let $\hat{\rho}$ be the minimum value of Eq. (15) for any set of directions e considered. Then an improved estimate of the degree of controllability is $\rho^{**} = \min(\rho^*, \hat{\rho})$, and $\rho^{**} \geq \rho$ can be made arbitrarily close to the true degree of controllability ρ by picking a sufficient number of directions e.

Computation of the Degree of Controllability in the Multiple Control Case

The previous section presented a procedure for generating an approximation ρ^{**} to the degree of controllability ρ in the case of a scalar control u. The procedure required the use of n carefully chosen directions in the state space, $b, Ab, \ldots, A^{n-1}b$, in generating the approximation to the recovery region, in order to insure that $\rho^* = 0$ (and hence $\rho^{**} = 0$) if, and only if, the system is uncontrollable. If the control vector is m dimensional with $m > 1$, it is no longer obvious how to obtain this property, since the columns of the $\mathcal{Q}$ matrix necessarily contain linearly dependent vectors. Instead, we will use the n linearly independent directions defined by the real eigenvectors and the real and imaginary parts of the complex eigenvectors of matrix A in Eq. (2), for the case where A has all distinct eigenvalues (the directions for the repeated eigenvalues case are given in the companion paper[3]). In the scalar control treatment above, ρ^* reduces to zero when the system is uncontrollable because the linear dependence of the vectors $b, Ab, \ldots, A^{n-1}b$ implies the collapse of at least one dimension of the parallelepiped. The vectors chosen here to handle multidimensional control problems will still cause ρ^* to reduce to zero properly under rather general conditions, but the collapse of the parallelepiped volume will occur without producing linear dependence of the chosen n directions.

The recovery region can be characterized by the following Lemma.

Lemma 1: Let A have all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with associated right and left eigenvectors being the column partitions of P and Q, respectively, where $Q^T = P^{-1}$ and

$$P = [p_1 \; p_2 \; \ldots \; p_n], \qquad Q = [q_1 \; q_2 \; \ldots \; q_n]$$

Then the state $x(0) = -\delta$ which is returned to the origin in time T using control $u(t)$ can be obtained from the fundamental equation

$$\delta = \sum_{j=1}^{n} p_j \left(\int_0^T e^{-\lambda_j t} q_j^T B u \, \mathrm{d}t \right) \tag{16}$$

and the recovery region is the set of all such δ generated by admissible control functions $|u_\beta(t)| \leq 1$; $t \in [0, T]$; $\beta = 1, 2, \ldots, m$.

Proof: In Eq. (3), $\phi(0,t) = \exp(-At)$, and since $P^{-1}AP = \Lambda$ where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$e^{-At} = P e^{-\Lambda t} P^{-1}$$

and

$$\delta = \int_0^T P e^{-\Lambda t} P^{-1} B u(t) \, \mathrm{d}t$$

which establishes Eq. (16) after partitioning the matrices. ∎

Let C be the set of j for which λ_j is real, and C^* be the set obtained by taking one value of j associated with each complex conjugate pair of eigenvalues, and write $p_j = p_j^R + i p_j^I$ for $j \in C_j^*$. The following theorem gives the parallelepiped approximation to the recovery region applicable to the multidimensional-control problem.

Theorem 1: Let A have all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. The recovery region $\mathcal{R}$ can be approximated by the (possibly degenerate) n-dimensional parallelepiped

$$\mathcal{R}^* = \left\{ x \mid x = \sum_{i=1}^{n} c_i \nu_i \; \forall \; c_i \ni |c_i| \leq 1 \right\} \tag{17}$$

where the vectors ν_i represent the following n vectors:

$$\left(\sum_{\beta=1}^{m}\int_0^T |f_{j\beta}|\mathrm{d}t\right)p_j \qquad j\in C \tag{18}$$

$$2\left(\sum_{\beta=1}^{m}\int_0^T |\mathrm{Re}(f_{j\beta})|\mathrm{d}t\right)p_j^R \qquad j\in C^* \tag{19}$$

$$2\left(\sum_{\beta=1}^{m}\int_0^T |\mathrm{Im}(f_{j\beta})|\mathrm{d}t\right)p_j^I \qquad j\in C^* \tag{20}$$

where $f_{j\beta}=e^{-\lambda_j t}q_j^T b_\beta$ and Re and Im denote real and imaginary parts. Furthermore, the approximate $\mathcal{R}^*$ is such that at least one point on every side (i.e., with one c_i set to 1 or -1) of the parallelepiped is contained in $\mathcal{R}$.

Proof: Consider a complex conjugate pair of eigenvalues λ_j and λ_i. Then p_j, q_j, $f_{j\beta}$ and p_i, q_i, $f_{i\beta}$ are complex conjugates, respectively, and Eq. (16) can be written in the form

$$\delta=\sum_{j\in C}\left(\sum_{\beta=1}^{m}\int_0^T f_{j\beta}u_\beta \mathrm{d}t\right)p_j + \sum_{j\in C^*}2\left(\sum_{\beta=1}^{m}\int_0^T \mathrm{Re}(f_{j\beta})u_\beta \mathrm{d}t\right)p_j^R - \sum_{j\in C^*}2\left(\sum_{\beta=1}^{m}\int_0^T \mathrm{Im}(f_{j\beta})u_\beta \mathrm{d}t\right)p_j^I \tag{21}$$

where the fact that δ must be real has been used.

It is proved in the companion paper, Ref. 3, that the vectors p_j, $j\in C$ and p_j^I, p_j^R for $j\in C^*$ form a set of n linearly independent vectors. The recovery region is the set of all δ generated by admissible controls, and this region forms a volume in n-dimensional space provided all integral coefficients of the directions in Eq. (21) are nonzero.

In order to maximize the coefficient of p_j in Eq. (21) (or equivalently maximize the component of δ along p_j, as seen in the usually nonorthogonal coordinates defined by the n linearly independent directions) the control components u_β must be chosen as $u_\beta(t)=\mathrm{sgn}(f_{j\beta})$ for $j\in C$ [where, for the sake of definiteness, $\mathrm{sgn}(f_{j\beta})$ is taken as zero if $f_{j\beta}=0$, so that u_β is zero when it has no influence on the p_j component of the state], and this produces Eq. (18). For directions p_j^R and p_j^I, the control functions are chosen as $u_\beta(t)=\mathrm{sgn}[\mathrm{Re}(f_{j\beta})]$ and $u_\beta(t)=-\mathrm{sgn}[\mathrm{Im}(f_{j\beta})]$, respectively, giving Eqs. (19) and (20). If each integral in Eq. (21) could be maximized independently, then the set of points $x(0)=-\delta$ generated would be the parallelepiped given in Eq. (17). Hence $\mathcal{R}\subset\mathcal{R}^*$. Since each integral in Eq. (21) can assume the corresponding maximum value given in Eqs. (18-20) as well as its negative, at least one point on each side of the parallelepiped $\mathcal{R}^*$ is in $\mathcal{R}$. ∎

The minimum perpendicular distance to a side of the parallelepiped $\mathcal{R}^*$ can be used as an approximation ρ^* of the degree of controllability ρ. Define the matrices

$$F=[\nu_1\ \nu_2\ \dots\ \nu_n], \qquad F^*=[\nu_1^*\ \nu_2^*\ \dots\ \nu_n^*]$$

$$\nu_j^*=\nu_j/\|\nu_j\| \qquad j=1,2,\dots,n \tag{22}$$

where the ν_j are defined in Eqs. (18-20), or in the event that the method of the previous section is used on a problem with a scalar control, the ν_j are calculated from Eqs. (10) and (13). For a controllable system all ν_j are nonzero, and it is shown in Ref. 3 that the ν_j are linearly independent. Each ν_j vector goes from the origin of the state space, or center of the parallelepiped volume, to the center of one of its sides. Any surface of the parallelepiped consists of edges that are parallel to $n-1$ of the ν vectors. Let d_j, corresponding to ν_j, represent the perpendicular distance from the origin to that surface for which no edge is parallel to ν_j. Then d_j is the component of ν_j normal to this surface, and can be calculated easily from the following theorem.

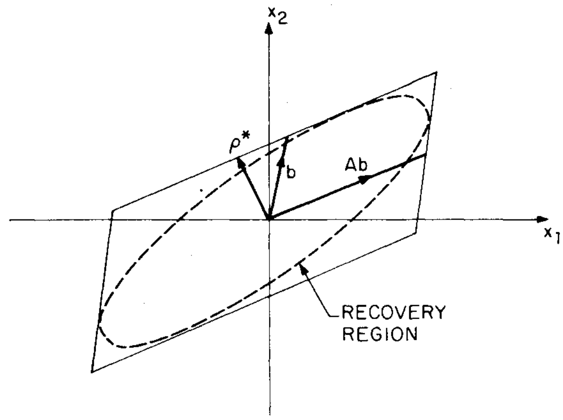

Fig. 2 Parallelepiped bound on the recovery region.

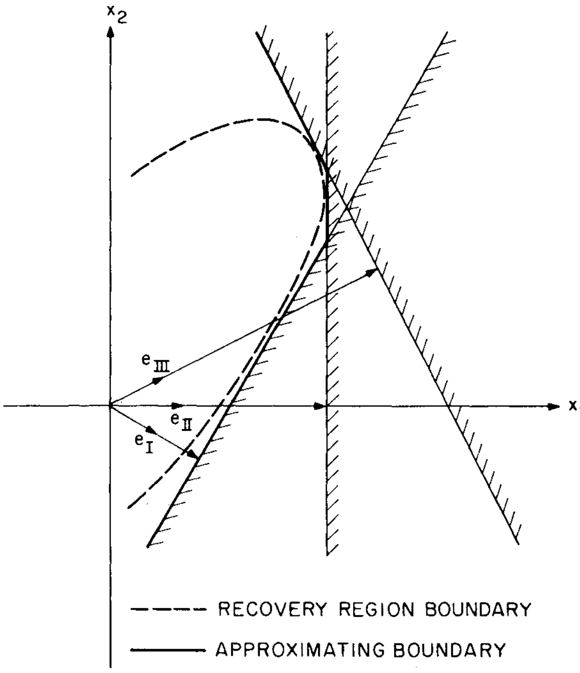

Fig. 3 Improving the approximation to the recovery region.

Theorem 2: Let the columns of F be linearly independent, and define the column vectors μ_j and μ_j^* as the partitions

$$(F)^{-1}=[\mu_1\ \mu_2\ \dots\ \mu_n]^T, \quad (F^*)^{-1}=[\mu_1^*\ \mu_2^*\ \dots\ \mu_n^*]^T \tag{23}$$

Then the normal distances d_j, $j=1,2,\dots,n$, to the surfaces of the n-dimensional parallelepiped prescribed by F are given by the two alternative equations:

$$d_j=1/\|\mu_j\| \tag{24}$$

$$d_j=\|\nu_j\|/\|\mu_j^*\| \tag{25}$$

Proof: Since $F^{-1}F$ is the identity matrix, $\mu_j^T\nu_k=\delta_{jk}$ where δ_{jk} is the Kronecker delta. Thus μ_j is perpendicular to all ν_k except ν_j, and hence is normal to the surface of the parallelepiped parallel to all ν_k for $k \neq j$. The desired d_j is the component of ν_j along this normal. Since $\mu_j^T \nu_j = 1$, the needed component of ν_j, $d_j = (\mu_j^T / \|\mu_j\|) \nu_j$, is given by Eq. (24).

The vector μ_j is an n-dimensional vector orthogonal to the $n-1$ dimensional space spanned by the ν_k for $k \neq j$. The vector μ_j^* from the appropriate row of $(F^*)^{-1}$ is orthogonal to the space spanned by $\nu_k / \|\nu_k\|$, which is the same space. Hence μ_j^* is a constant multiple of μ_j, which is given by $\mu_j = \mu_j^* / \|\nu_j\|$, since $\mu_j^T \nu_j = 1$ and $\mu_j^{*T} \nu_j^* = 1$. This establishes Eq. (25). ∎

The computation of the d_j from F^* using Eq. (25) is to be preferred. For systems with a poor degree of controllability the magnitudes of one or more of the ν_j must be small, and this could make the computation of F^{-1} difficult, whereas the computation of $(F^*)^{-1}$ is independent of these magnitudes. Note that F^* can be calculated directly from the eigenvectors p_j, $j \epsilon C$ and p_j^R, p_j^I for $j \epsilon C^*$ without regard for the magnitudes of the parallelepiped semixes.

The approximation ρ^* of the degree of controllability ρ is then given as

$$\rho^* = \min_j d_j \tag{26}$$

As before, it is an upper bound on the true degree of controllability, but it can be improved, and made arbitrarily tight, by considering other directions e in the state space. Assuming all eigenvalues λ_j of A are real, Eq. (16) implies that the m control histories which maximize the e component of $x(0)$ are

$$u_\beta(t) = -\operatorname{sgn}\left[\sum_{j=1}^{n} (e^T p_j) f_{j\beta}\right]$$

and the value of this component is

$$\max[e^T x(0)] = \sum_{\beta=1}^{m} \int_0^T \left| \sum_{j=1}^{n} (e^T p_j) f_{j\beta} \right| \mathrm{d}t$$

Similar results are obtained when complex eigenvalues are present by using Eq. (21). If the maximized components of $x(0)$ are obtained for a family of directions e, and the minimum value obtained is $\hat{\rho}$, then the approximation $\rho^{**} (\rho^* \geq \rho^{**} \geq \rho)$ to the degree of controllability is

$$\rho^{**} = \min(\rho^*, \hat{\rho}) \tag{27}$$

For an uncontrollable system, the degree of controllability ρ is zero by definition. If ρ^* (and hence ρ^{**}) is to be a useful approximation to ρ, it should have the property that ρ^* is also zero for an uncontrollable system.

Theorem 3: Let all eigenvalues of A be distinct. Then $\rho^* = 0$ if and only if $\rho = 0$. That is, the approximate degree of controllability based on the minimum distance to the surface of the parallelepiped $\mathcal{R}^*$ is zero if, and only if, the system is uncontrollable.

Proof: Assume first that the system (A,B) is controllable. A system for which all eigenvalues of A are distinct is controllable if, and only if, the control coefficient matrix of the state equation when expressed in diagonalized form has the property that there is at least one nonzero element in every row. Hence, in a controllable system, there exists a β for each j such that $q_j^T b_\beta \neq 0$. Then, at least one of the terms $|f_{j\beta}|$ in the summation of Eq. (18) is nonzero. Similarly, the integral in Eq. (19) can be written as

$$\int_0^T |e^{-\sigma_j t} [q_j^R \cos\omega_j t + q_j^I \sin\omega_j t]^T b_\beta| \mathrm{d}t$$

where $\lambda_j = \sigma_j + i\omega_j$; $q_j = q_j^R + iq_j^I$; and $b_\beta^T q_j^R$ or $b_\beta^T q_j^I \neq 0$ for at least one β. Due to linear independence of the cosine and sine functions, the integrand for this β cannot be zero over any nonzero interval $[0,T]$. Therefore, for a controllable system, for each $j \epsilon C^*$, Eq. (19) defines a nonzero semiaxis. Analogous arguments prove the same result for Eq. (20).

The semiaxis directions defined by p_j, $j \epsilon C$, and p_j^R, p_j^I for $j \epsilon C^*$ are proved to be linearly independent in Ref. 3. Since F is composed of nonzero multiples of these vectors, F^{-1} exists and is nonsingular, and each $d_j = 1/\|\mu_j\|$ is strictly positive. Therefore, if the system is controllable ($\rho > 0$), then $\rho^* > 0$.

To complete the proof, it suffices to show that if $\rho = 0$ then $\rho^* = 0$. For an uncontrollable system with distinct eigenvalues, there exists some j for which $q_j^T b_\beta = 0$ for all $\beta = 1,2,\ldots,m$, which implies that

$$\sum_{\beta=1}^{m} |f_{j\beta}| = 0 \quad \text{if } j \epsilon C,$$

and

$$\sum_{\beta=1}^{m} |\mathrm{Re}(f_{j\beta})| = \sum_{\beta=1}^{m} |\mathrm{Im}(f_{j\beta})| = 0 \quad \text{if } j \epsilon C^*$$

Therefore, at least one of the semiaxes of the parallelepiped is zero, the shortest distance from the origin to the surface of the parallelepiped is zero, and thus $\rho^* = 0$ from its definition. ∎

It should be noted that a certain type of superposition holds for the recovery region, but that this type of superposition does not hold for the degree of controllability itself.

Superposition Properties: Consider system (2) with m independent control variables u_β, $\beta = 1,2,\ldots,m$.

1) Let e be any direction in the state space. The maximum component along direction e for all points in the recovery region, is equal to the sum of m maximum components along this direction obtained using each control u_β alone.

2) The length of each semiaxis of $\mathcal{R}^*$, the parallelepiped approximation to the recovery region, is equal to the sum of the m corresponding semiaxis lengths obtained using each control u_β alone.

To show these properties it is best to work with δ as expressed in terms of real valued quantities in Eq. (21). Rearranging the equation gives

$$\delta(u) = \sum_{\beta=1}^{m} \delta_\beta(u_\beta)$$

$$\delta_\beta(u_\beta) = \int_0^T \Big\{ \sum_{j \epsilon C} f_{j\beta} p_j + 2 \sum_{j \epsilon C^*} \mathrm{Re}(f_{j\beta}) p_j^R - 2 \sum_{j \epsilon C^*} \mathrm{Im}(f_{j\beta}) p_j^I \Big\} u_\beta \mathrm{d}t \tag{28}$$

and hence

$$\max_{u_1, u_2, \ldots, u_m} [e^T \delta(u)] = \sum_{\beta=1}^{m} \max_{u_\beta} e^T \delta_\beta(u_\beta)$$

which establishes Property 1 above. The semiaxes of $\mathcal{R}^*$ are given by the maximum values of the coefficients of p_j, p_j^R, and p_j^I in Eq. (21), and analogous arguments apply to these coefficients to establish Property 2.

Specialization of the Degree of Controllability Criterion to Modal Coordinates

Consider a flexible spacecraft with dynamic equations expressed in terms of spacecraft normal modes, and consider the shape control problem. (In order to treat the combination of shape and attitude control, see Ref. 3 for the manner of handling the repeated roots associated with rigid body modes.) Then the equations are

$$\ddot{\eta}_j + 2\zeta_j\omega_j\dot{\eta}_j + \omega_j^2\eta_j = \Gamma_j^T u^* \qquad j = 1,2,\ldots,(n/2) \tag{29}$$

The Γ_j and u^* are defined so that each $|u_i^*|$ is bounded by the unnormalized actuator strength k_i. Let the numbers N_j represent the relative importance we assign to the control of the η_j's. We must also specify the importance we assign to the control of $\dot{\eta}_j$ in the state space which will be generated. Now define the state variables as follows

$$x_{2j-1} \triangleq \eta_j/N_j \qquad x_{2j} \triangleq \dot{\eta}_j/\bar{N}_j \tag{30}$$

so that all unit deviations from the origin of the state space will be considered equally serious. Then

$$\begin{bmatrix} \dot{x}_{2j-1} \\ \dot{x}_{2j} \end{bmatrix} = \begin{bmatrix} 0 & \bar{N}_j/N_j \\ -\omega_j^2 N_j/\bar{N}_j & -2\zeta_j\omega_j \end{bmatrix} \begin{bmatrix} x_{2j-1} \\ x_{2j} \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma_j^T/\bar{N}_j \end{bmatrix} u^* \tag{31}$$

Let the coefficient matrices be A_j and B_j. Then the normalized system equation (2) has coefficient matrices A and B, and eigenvector matrix P, which can be partitioned as follows:

$$\begin{aligned} A &= \mathrm{diag}[A_1, A_2, \ldots, A_{n/2}] \\ B &= [B_1^T, B_2^T, \ldots, B_{n/2}^T]^T \\ P &= \mathrm{diag}[P_1, P_2, \ldots, P_{n/2}] \\ P^{-1} &= \mathrm{diag}[P_1^{-1}, P_2^{-1}, \ldots, P_{n/2}^{-1}] \end{aligned} \tag{32}$$

where

$$P_j = \begin{bmatrix} 1 & 1 \\ \lambda_{2j-1}N_j/\bar{N}_j & \lambda_{2j}N_j/\bar{N}_j \end{bmatrix}$$

$$P_j^{-1} = \frac{1}{\lambda_{2j} - \lambda_{2j-1}} \begin{bmatrix} \lambda_{2j} & -\bar{N}_j/N_j \\ -\lambda_{2j-1} & \bar{N}_j/N_j \end{bmatrix}$$

$$\lambda_{2j-1} = \omega_j(-\zeta_j + i\sqrt{1-\zeta_j^2})$$

$$\lambda_{2j} = \omega_j(-\zeta_j - i\sqrt{1-\zeta_j^2})$$

Using the rows of P^{-1} and the columns of B to form $f_{j\beta}$ from Theorem 1 gives the following expression for the integral in Eq. (19)

$$\left(\int_0^T \sum_{\beta=1}^m \frac{|\Gamma_{j\beta}|k_\beta}{N_j\omega_j\sqrt{1-\zeta_j^2}} e^{\zeta_j\omega_j t} |\sin(\sqrt{1-\zeta_j^2}\,\omega_j t)|\,\mathrm{d}t\right) p_{2j}^R \tag{33}$$

This expression can be simplified considerably if we assume that the damping is such that the exponential will not change significantly during one oscillation of the sine wave, and that T is long compared with the period of the sine wave (so that the effect of partial completion of the final period of oscillation of the sine wave is negligible). Then the absolute value of the sine can be replaced by $2/\pi$, its average over a period, and Eq. (33) becomes

$$\frac{2}{\pi}\frac{\|\Gamma_j\|_k}{N_j\omega_j}\left(\frac{e^{\zeta_j\omega_j T} - 1}{\zeta_j\omega_j\sqrt{1-\zeta_j^2}}\right) p_{2j}^R \tag{34}$$

where the weighted norm is defined by

$$\|\Gamma_j\|_k \triangleq \sum_{\beta=1}^m |\Gamma_{j\beta}|k_\beta \tag{35}$$

The analogous calculation for Eq. (20) gives the same result with p_{2j}^R replaced by p_{2j}^I. Together these vectors form the set $v_1, v_2, \ldots, v_n$, which are the columns of F.

The p_{2j}^R and p_{2j}^I are the real and imaginary parts of the appropriate column of P. Looking only at the appropriate partition of F, call it F_j, we have

$$F_j = \sigma \begin{bmatrix} 1 & 0 \\ -\zeta_j N_j\omega_j/\bar{N}_j & -\sqrt{1-\zeta_j^2}\,N_j\omega_j/\bar{N}_j \end{bmatrix}$$

$$F_j^{-1} = \frac{1}{\sigma\sqrt{1-\zeta_j^2}} \begin{bmatrix} \sqrt{1-\zeta_j^2} & 0 \\ -\zeta_j & -\bar{N}_j/(N_j\omega_j) \end{bmatrix}$$

where σ is the coefficient of p_{2j}^R in Eq. (34). The associated values of d_j from Theorem 2 result from taking the magnitude of the rows of F_j^{-1} and are given by

$$d_{2j-1} = \sigma, \qquad d_{2j} = \sigma\left(\frac{1-\zeta_j^2}{\bar{N}_j^2/(N_j\omega_j)^2 + \zeta_j^2}\right)^{1/2}$$

From Eq. (26) the approximate degree of controllability ρ^* is

$$\rho^* = \min_j \frac{2}{\pi}\left\{\frac{\|\Gamma_j\|_k}{N_j\omega_j}\left(\frac{e^{\zeta_j\omega_j T} - 1}{\zeta_j\omega_j}\right)\min\left[(1-\zeta_j^2)^{-1/2},\ (\bar{N}_j^2/(N_j\omega_j)^2 + \zeta_j^2)^{-1/2}\right]\right\} \tag{36}$$

Note that when damping is present, the recovery region expands with T in such a way that ρ^* grows exponentially. An important special case is that of a lightly damped system, in which case the normalizations N_j and $\bar{N}_j$ should be related by $\bar{N}_j = N_j\omega_j$, since if η_j is $\sin(\omega_j t + \phi)$, then $\dot{\eta}_j$ is approximately $\omega_j\cos(\omega_j t + \phi)$. Equation (36) then becomes

$$\rho^* = \min_j \frac{2}{\pi}\left\{\frac{\|\Gamma_j\|_k}{N_j\omega_j}\left[\frac{e^{\zeta_j\omega_j T} - 1}{\zeta_j\omega_j(1+\zeta_j^2)^{1/2}}\right]\right\} \tag{37}$$

If there is no damping present, which is a reasonable representation of the expected very large flexible spacecraft, then the approximate degree of controllability further reduces to the very simple and useful form

$$\rho^* = (2T/\pi)\min_j[\|\Gamma_j\|_k/(N_j\omega_j)] \tag{38}$$

Examination of the form of this result shows that the approximate degree of controllability for this type of system is simply the weighted minimum of the norms [as defined in Eq. (35) to include actuator strength] of the rows of the input influence matrix when the system equations are represented in modal form. Thus the result can be thought of as a generalization of one of the simple tests for controllability by examining the rows of the input influence matrix when the system is in diagonal form. However, such a simple interpretation only applies in the case of second-order modal systems without damping.

Note that if one is not particularly interested in controlling any specific mode, the corresponding N_j will be very small, and hence that mode has no influence on the approximate degree of controllability ρ^*.

An important property of the result, Eq. (38), is that ρ^* increases linearly with time, making T simply a scale factor. As stated earlier, the definition of the degree of controllability must depend on T. However, it is important to note that the scale factor nature of the dependence means that in the use of the approximate degree of controllability to place actuators, the parameter T will have no influence, and can be dropped, whenever the T of interest is large compared with the longest period of modal oscillation in the system.

Conclusions and Discussion

This paper represents the first part of a two-part work, the second part of which is published elsewhere as Ref. 3. The reader is referred to Ref. 3 for a complete presentation of the approximate recovery region and the appropriate spanning vectors for determination of the approximate degree of controllability. This general treatment handles repeated eigenvalues which always occur from rigid body modes, and which occur in other modes for certain spacecraft, for example, for one with the symmetries of a square plate. Examples are also presented in Ref. 3 which give some indications of the implications of degree of controllability concepts on optimal actuator locations.

This paper, together with Ref. 3, is a seminal work that has spawned many related publications which are currently available, at least in conference proceedings form.[5-15] A discussion of how these references are related is instructive and puts the work presented here in perspective. This is most easily done by examining the five points presented in the Definition of the Degree of Controllability section as being required of any definition. Item 4 said that the definition must standardize or restrict the control effort in some way. Examination of the approach here shows that this is accomplished in two ways. First, the magnitudes of each control component have been limited by saturation constraints, $|u_i| \leq 1$. The second part is somewhat more subtle. The recovery region defined here gives the set of initial conditions that can be returned to the origin in time T, and hence the boundary of this region is composed of initial states for which time T is the minimum possible time. Hence, implicit in the definition is the use of a time optimal control law. Once this is recognized, it is clear that there are many possibilities. Reference 5 develops the energy optimal degree of controllability concepts, while Refs. 6 and 7 consider the fuel optimal degree of controllability concept. Reference 8 develops still other concepts in which certain possibilities are unified within a framework based on defining gain measures. The fundamental structures underlying all such definitions is developed in detail in Ref. 9, which also includes a discussion of the undesirable properties associated with attempting to use a quadratic cost control law. Reference 10 moves out of the framework developed here, which puts emphasis on response to initial conditions, and considers concepts of the degree of disturbance rejection.

The fifth point required of any definition was that the control objective must be reflected in the definition. This work as well as most of the references are concerned with the regulator problem of returning the state to the origin after an initial disturbance. When a given maneuver is to be performed repeatedly, this alters the appropriate definition of the degree of controllability, as discussed in Refs. 11 and 5. If one is interested in tracking problems where the maneuver is not known a priori, or if one is interested in model reduction applications, it may be more natural to consider the use of a reachable region rather than a recovery region.[8,9]

The third item required of any definition was that dependence on total time T must be present. Although we recognize that this is so, it represents an extra parameter which must be specified before the degree of controllability can be determined and used. Reference 12 discusses methods of choosing T as well as the normalization of the state space, and illustrates the associated implications for actuator placement.

The approach to computing the degree of controllability presented here gives an approximation which is an upper bound. A lower bound can be generated and made arbitrarily good by employing a discretized form of the state equations, as discussed in Ref. 13. However, the upper bound approximation has been shown here to be extremely easy to generate for modal systems with sufficiently large T, and the accuracy of the upper bound approximation, studied in Ref. 14, is found to be very good, in fact, exact twice every period, for such harmonic oscillator equations. Rigid body modes on the other hand are best treated using the exact solution given in Ref. 14.

The use of degree of controllability for optimizing actuator locations is studied in Refs. 3, 7, 12, and 15.

Controllability and observability are dual concepts, and hence once one has generated a degree of controllability, there should be an associated degree of observability concept which could be of use in choosing good locations for sensors. Degree of observability is studied in Ref. 9 as well as in Ref. 8.

For purposes of model reduction, Moore[16] has applied singular value decomposition of the controllability and the observability Gramians, and developed concepts of balanced forms. The structure developed here and in Refs. 1-15 form a parallel development that contains controllability and observability Gramians as a special case associated with energy optimality.[5] Balanced forms for other possibilities are discussed in Refs. 8 and 9.

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